3. INNER PRODUCT SPACES

§3.1. Definition

So far we have studied abstract vector spaces. These are a generalisation of the geometric spaces $\mathbf{R}^2$ and $\mathbf{R}^3$. But these have more structure than just that of a vector space. In R^2 and R^3 we have the concepts of lengths and angles. In those spaces we use the dot product for this purpose, but the dot product only makes sense when we have components. In the absence of components we introduce something called an **inner product** to play the role of the dot product. We consider only vector spaces over C, or some subfield of C, such as R.

An **inner product space** is a vector space V over **C** together with a function (called an inner product) that associates with every pair of vectors in V a complex number $\langle u \mid v \rangle$ such that:

(1) $\langle v|u\rangle = \overline{\langle u|v\rangle}$ for all u, v ∈ V;

(2) $\langle u+v|w\rangle = \langle u|w\rangle + \langle v|w\rangle$ for all u, v, w ∈ V;

(3) $\langle \lambda u|v\rangle = \lambda\langle u|v\rangle$ for all u, v ∈ V and all λ ∈ **C**;

(4) $\langle v|v\rangle$ is real and ≥ 0 for all v ∈ V;

(5) $\langle v|v\rangle$ = 0 if and only if v = 0.

These are known as the axioms for an inner product space (along with the usual vector space axioms).

A **Euclidean space** is a vector space over **R**, where $\langle u|v\rangle \in \mathbf{R}$ for all u, v and where the above five axioms hold. In this case we can simplify the axioms slightly:

(1) $\langle v|u\rangle = \langle u|v\rangle$ for all u, v ∈ V;

(2) $\langle u+v|w\rangle = \langle u|w\rangle + \langle v|w\rangle$ for all u, v, w ∈ V;

(3) $\langle \lambda u|v\rangle = \lambda\langle u|v\rangle$ for all u, v ∈ V and all λ ∈ **C**;

(4) $\langle v|v\rangle \geq 0$ for all v ∈ V;

(5) $\langle v|v\rangle$ = 0 if and only if v = 0.

Example 1: Take V = $\mathbf{R}^n$ as a vector space over **R** and define $\langle u|v\rangle = u_1v_1 + \ldots + u_nv_n$ where $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ (the usual dot product). This makes $\mathbf{R}^n$ into a Euclidean space. When n = 2 we can interpret this geometrically as the real Euclidean plane. When n = 3 this is the usual Euclidean space.

Example 2: Take V = $\mathbf{C}^n$ as a vector space over **C** and define $\langle u|v\rangle = u_1\overline{v_1} + \ldots + u_n\overline{v_n}$ where $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$.

Example 3: Take V = $M_n(\mathbf{R})$, the space of n×n matrices over **R** where $\langle A \mid B\rangle = \text{trace}(A^TB)$. Note, this becomes the usual dot product if we consider an n × n matrix as a vector with n^2 components, since $\text{trace}(A^TB) = \sum_{i,j=1}^{n} a_{ij}b_{ij}$ if $A = (a_{ij})$ and $B = (b_{ij})$.

Example 4: Show that $\mathbf{R}^2$ can be made into a Euclidean space by defining

$$\langle \mathbf{u_1} \mid \mathbf{u_2}\rangle = 5x_1x_2 - x_1y_2 - x_2y_1 + 5y_1y_2$$

when $\mathbf{u_1} = (x_1, y_1)$ and $\mathbf{u_2} = (x_2, y_2)$.

Solution: We check the five axioms.

(1) $\langle \mathbf{u_2} \mid \mathbf{u_1}\rangle = 5x_2x_1 - x_2y_1 - x_1y_2 + 5y_2y_1 = \langle \mathbf{u_1} \mid \mathbf{u_2}\rangle$.
(2) If $\mathbf{u_3} = (x_3, y_3)$ then $\langle u_1 + u_2 \mid u_3\rangle = 5(x_1 + x_2)x_3 - x_3(y_1 + y_2) - (x_1 + x_2)y_3 + 5(y_1 + y_2)y_3$

$= 5x_1x_3 + 5x_2x_3 - x_3y_1 - x_3y_2 - x_1y_3 - x_2y_3 + 5y_1y_3 + 5y_2y_3$

$= (5x_1x_3 - x_3y_1 - x_1y_3 + 5y_1y_3) + (5x_2x_3 - x_3y_2 - x_2y_3 + 5y_2y_3)$

$= \langle \mathbf{u_1} \mid \mathbf{u_3}\rangle + \langle \mathbf{u_2} \mid \mathbf{u_3}\rangle$.

(3) $\langle \lambda\mathbf{u_1} \mid \mathbf{u_2}\rangle = 5(\lambda x_1)x_2 - x_2(\lambda y_1) - (\lambda x_1)y_2 + 5(\lambda y_1)y_2$

$= \lambda[5x_1x_2 - x_2y_1 - x_1y_2 + 5y_1y_2]$

$= \lambda\langle \mathbf{u_1} \mid \mathbf{u_3}\rangle$.

(4) If $\mathbf{v} = (x, y)$ then $\langle v \mid v\rangle = 5x^2 - 2xy + 5y^2$

$= 5(x^2 - 2xy/5 + y^2)$

$= 5(x - y/5)^2 + 24y^2/25$

≥ 0 for all x, y.

(5) $\langle \mathbf{v} \mid \mathbf{v}\rangle = 0$ if and only if $x = y/5$ and $y = 0$, that is, if and only if $\mathbf{v} = 0$.

Now we move to a rather different sort of inner product, but one that still satisfies tha above axioms. Inner product spaces of this type are very important in mathematics.

Example 5: Take V to be the space of continuous functions of a real variable and define

$$\langle u(x) | v(x)\rangle = \int_0^{2\pi} u(x)v(x)\,dx$$

NOTE: Axioms (2), (3) show that the function u $\to \langle u | v\rangle$ is a linear transformation for a fixed v.

However $u \to \langle v|u\rangle$ is not linear since $\langle v|\lambda u\rangle = \overline{\langle \lambda u|v\rangle} = \overline{\lambda\langle u|v\rangle} = \overline{\lambda}\ \overline{\langle u|v\rangle} = \overline{\lambda}\langle v|u\rangle$.

§3.2. Lengths and Distances

The **length** of a vector in an inner product space is defined by:

$$|v| = \sqrt{\langle v|v\rangle}\ .$$

(Remember that $\langle v|v\rangle$ is real and non-negative. The square root is the non-negative one.)

So the zero vector is the only one with zero length. All other vectors in an inner product space have positive length.

Example 6: In $\mathbf{R}^3$, with the dot product as inner product, the length of (x, y, z) is $\sqrt{x^2 + y^2 + z^2}$.
Example 6: If V is the space of continuous functions of a real variable and

$\langle u(x)|v(x)\rangle = \int_0^1 u(x)v(x)\,dx = \int_0^1 f(x)g(x)\,dx$ then the length of $f(x) = x^2$ is $\sqrt{\int_0^1 x^4\,dx} = \dfrac{1}{\sqrt{5}}$.

The following properties of length are easily proved.

Theorem 1: For all vectors u, v and all scalars λ:

(1) $|\lambda v| = |\lambda|.|v|$;
(2) $|v| \geq 0$;
(3) $|v| = 0$ if and only if $v = 0$.

Theorem 2 (Cauchy Schwarz Inequality):

$$|\langle u \mid v\rangle| \le |u|.|v|.$$

$$\text{Equality holds if and only if } u = \frac{\langle u \mid v\rangle\, v}{|v|^2}.$$

Proof: Let $d = \dfrac{\langle u \mid v\rangle}{|v|^2}$.

Now $|u - dv|^2 = \langle u - dv \mid u - dv\rangle$

$= \langle u \mid u\rangle - d\,\langle v \mid u\rangle - \bar{d}\,\langle u \mid v\rangle + d\,\bar{d}\,\langle v \mid v\rangle$

$= |u|^2 - 2d\,\bar{d}\,|v|^2 + d\,\bar{d}\,|v|^2$

$= |u|^2 - |d|^2|v|^2$

$= |u|^2 - \dfrac{|\langle u \mid v\rangle|^2}{|v|^2}$

Since $|u - dv|^2 \ge 0$, $|u|^2|v|^2 \ge |\langle u \mid v\rangle|^2$.

Example 7: In $\mathbf{R}^n$ we have $\left(\sum x_i y_i\right)^2 \le \left(\sum x_i^{\,2}\right)\left(\sum y_i^{\,2}\right)$.

Example 8: $\left(\displaystyle\int_0^1 f(x)g(x)\,dx\right)^2 \le \left(\displaystyle\int_0^1 f(x)^2\,dx\right)\left(\displaystyle\int_0^1 g(x)^2\,dx\right)$.

The Triangle Inequality in the Euclidean plane states no side of a triangle can be longer than the sum of the other two sides. It is usually proved geometrically, or appealing to the principle that the shortest distance between two points is a straight line. In a general inner product space we must prove it from the axioms.

Theorem 3 (Triangle Inequality): For all vectors u, v: $|u + v| \le |u| + |v|$.

Proof: $|u + v|^2 = \langle u + v \mid u + v\rangle$

$= \langle u \mid u\rangle + \langle v \mid v\rangle + \langle u \mid v\rangle + \langle v \mid u\rangle$

$= |u|^2 + |v|^2 + 2\mathrm{Re}(\langle u \mid v\rangle)$

$\le |u|^2 + |v|^2 + 2|\langle u \mid v\rangle|$

$\le |u|^2 + |v|^2 + 2|u|.|v|$

$\le (|u| + |v|)^2$

So $|u + v| \le |u| + |v|$.

We define the **distance** between two vectors u, v to be $|u - v|$. The distance version of the Triangle Inequality is as follows. If u, v, w are the vertices of a triangle in an inner product space V then $|u - w| \le |u - v| + |v - w|$. It follows from the length version as $u - w = (u - v) + (v - w)$. If we take u, v, w to be vertices of a triangle in the Euclidean plane this gives the geometric version of the Triangle Inequality.

§3.3. Orthogonality

It is not possible to define angles in a general inner product space, because inner products need not be real. But in any Euclidean space we can define these geometrical concepts even if the vectors have no obvious geometric significance.

Now we can use the Cauchy Schwarz inequality to define the angle between vectors. If u, v are non-zero vectors the **angle** between them is defined to be $\cos^{-1}\left(\frac{\langle u \mid v\rangle}{|u|.|v|}\right)$. The Cauchy Schwarz inequality ensures that $\frac{\langle u \mid v\rangle}{|u|.|v|}$ lies between −1 and 1. The angle between the vectors is π/2 if and only if ⟨u | v⟩ = 0.

Example 9: Suppose we define the inner product between two continuous functions by $\langle u|v\rangle = \int_0^{\pi/2} u(x)v(x)dx$. If u(x) = sin x and v(x) = x find the angle, between them in degrees.

Solution: $\langle u|v\rangle = \int_0^{\pi/2} x\sin x\,dx$

$$= [\sin x - x\cos x]_0^{\pi/2} \quad \text{(integrating by parts)}$$

$$= 1.$$

$$\langle u(x)|u(x)\rangle = \int_0^{\pi/2} \sin^2 x\,dx$$

$$= \int_0^{\pi/2}\left(\frac{1-\cos 2x}{2}\right)dx$$

$$= \left[\frac{x}{2}+\frac{1}{4}\sin 2x\right]_0^{\pi/2}$$

$$= \frac{\pi}{4} \text{ which is approximately 0.7854.}$$

Hence |u(x)| is approximately 0.8862.

$$\langle v(x)|v(x)\rangle = \int_0^{\pi/2} x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^{\pi/2}$$

$$= \frac{\pi^3}{24} \text{ which is approximately 1.2919.}$$

Hence |v(x)} is approximately 1.1366.

So if the angle (in degrees) between these two functions is θ then

$\cos\theta \approx \frac{1}{0.8862*1.1366} \approx \frac{1}{1.00725} \approx 0.9928.$

Hence θ ≈ 6.8796°.

NOTE: Measuring the angle between two functions in degrees is rather useless and is done here only as a curiosity. By far the major application of angles in function spaces is to orthogonality. This is a concept that is meaningful for all inner product spaces, not just Euclidean ones.

Two vectors in an inner product space are **orthogonal** if their inner product is zero. The same definition applies to Euclidean spaces, where angles are defined and there orthogonality means that either the angle between the vectors is π/2 or one of the vectors is zero. So orthogonality is slightly more general than perpendicularity.

A vector v in an inner product space is a **unit vector** if |v| = 1.

We define a set of vectors to be **orthonormal** if they are all unit vectors and each one is orthogonal to each of the others. An **orthonormal basis** is simply a basis that is orthonormal. Note that there is no such thing as an "orthonormal vector". The property applies to a whole set of vectors, not to an individual vector.

Theorem 4: An orthonormal set of vectors $\{v_1, \ldots, v_n\}$ is linearly independent.
Proof: Suppose $\lambda_1 v_1 + \ldots + \lambda_n v_n = 0$.
Then $\langle \lambda_1 v_1 + \ldots + \lambda_n v_n \mid v_r\rangle = 0$ for each r.
But $\langle \lambda_1 v_1 + \ldots + \lambda_n v_n \mid v_r\rangle = \lambda_1\langle v_1 \mid v_r\rangle + \ldots + \lambda_n\langle v_n \mid v_r\rangle$
$= \lambda_r\langle v_r \mid v_r\rangle$ since v_r is orthogonal to the other vectors in the set
$= \lambda_r$ since $\langle v_r \mid v_r\rangle = |v_r|^2 = 1$.
Hence each $\lambda_r = 0$.

Because of the above theorem, if we want to show that a set of vectors is an orthonormal basis we need only show that it is orthonormal and that it spans the space. Linear independences come free.

Another important consequence of the above theorem is that it is very easy to find the coordinates of a vector relative to an orthonormal basis.

Theorem 5: If $\alpha_1, \alpha_2, \ldots, \alpha_n$ is an orthogonal basis for the inner product V, and $v \in V$, then

$$\left[\frac{v}{\alpha}\right] = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix} \text{ where } x_i = \frac{\langle v \mid \alpha_i\rangle}{|\alpha_i|^2}.$$

Proof: Let $v = x_1\alpha_1 + x_2\alpha_2 + \ldots + x_n\alpha_n$.
Then $\langle v \mid \alpha_i\rangle = \sum_j x_j \langle \alpha_i \mid \alpha_j\rangle$
$= x_j \langle \alpha_j \mid \alpha_j\rangle$ since the α_i are mutually orthogonal

Example 10: Consider $\mathbf{R}^3$ as an inner product space with the usual inner product.
Show that the set $\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right), \left(-\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right), \left(-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)$ is an orthonormal basis for $\mathbf{R}^3$.
Solution: They are clearly mutually orthogonal and, since $\frac{1}{9} + \frac{4}{9} + \frac{4}{9} = 1$ they are all unit vectors. Hence they are linearly independent and so span a 3-dimensional subspace of $\mathbf{R}^3$. Clearly this must be the whole of $\mathbf{R}^3$.

Example 11: Find the coordinates of (3, 4, 5) relative to the above orthonormal basis.
Solution:
$\langle (3, 4, 5) \mid (1/3, 2/3, 2/3)\rangle = 7$
$\langle (3, 4, 5) \mid (-2/3, 2/3, -1/3)\rangle = -1$
$\langle (3, 4, 5) \mid (-2/3, -1/3, 2/3)\rangle = 0$.
Hence the coordinates are (7, −1, 0). In other words, $(3, 4, 5) = 7\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) - \left(-\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)$.

Example 12: In $\mathbf{C}^2$ as an inner product space with the inner product

$$\langle (x_1, y_1) \mid (x_2, y_2)\rangle = x_1 \overline{x_2} + y_1 \overline{y_2}$$

show that the vectors $(2 - i, 3 - 4i)$ and $(3 - 4i, -2/5 + 11/5\, i)$ are orthogonal. Use them to find an orthonormal basis for $\mathbf{C}^2$.

Solution: $(2 - i)(3 + 4i) + (3 - 4i)(-2/5 - 11/5\, i) = 10 + 5i - 10 - 5i = 0$.
$|(2 - i, 3 - 4i)| = \sqrt{5 + 25} = \sqrt{30}$ and
$|(3 - 4i, -2/5 + 11/5\, i)| = \sqrt{5 + 5} = \sqrt{10}$.
Hence $\left\{\frac{1}{\sqrt{30}}(2 - i, 3 - 4i),\ \frac{1}{\sqrt{10}}(3 - 4i, -2/5 + 11/5\, i)\right\}$ is an orthonormal basis.

Theorem 6 (Gram-Schmidt):
Every finite-dimensional inner product space V has an orthonormal basis.
Proof: We prove this by induction on the dimension of V.
If dim(V) = 0 then the empty set is an orthonormal basis.
Suppose that every vector space of dimension n has an orthonormal basis of V and suppose that V is a vector space of dimension n + 1.
Let $\{u_1, \ldots, u_{n+1}\}$ be a basis for V and let $U = \langle v_1, \ldots, v_n\rangle$.
By the induction hypothesis U has an orthonormal basis $v_1, \ldots, v_n$.
Define $w = u_{n+1} - \langle u_{n+1} \mid v_1\rangle v_1 - \ldots - \langle u_{n+1} \mid v_n\rangle v_n$.
Then for each i, $\langle w \mid v_i\rangle = \langle u_{n+1} \mid v_i\rangle - \langle u_{n+1} \mid v_i\rangle\langle v_i \mid v_i\rangle$, since $\langle v_j \mid v_i\rangle = 0$ when $i \neq j$
$= \langle u_{n+1} \mid v_i\rangle - \langle v \mid v_i\rangle$ since $\langle v_i \mid v_i\rangle = 1$
$= 0$.
Hence w is orthogonal to each of the v_i. But it may not be a unit vector, but it is non-zero. So we may divide it by its length without affecting orthogonality.
So we define $v_{n+1} = \frac{1}{|w|} w$ and so obtain an orthonormal basis $\{v_1, \ldots, v_n, v_{n+1}\}$ for V.

In practice it is inconvenient to normalise the vectors (divide by their length) as we go, because we will have to carry these lengths along into our subsequent calculations. It's much easier to produce an orthogonal basis and then to normalise at the end.

GRAM-SCHMIDT ALGORITHM

$\{u_1, u_2, \ldots, u_n\}$ is a given basis;
$\{v_1, v_2, \ldots, v_n\}$ is an orthogonal basis;
$\{w_1, w_2, \ldots, w_n\}$ is an orthonormal basis.

(1) LET $\mathbf{v_1 = u_1}$.
(2) FOR r = 2 TO n, LET
$$\mathbf{v_r = u_r - \frac{\langle u_r|v_1\rangle}{|v_1|^2} v_1 - \frac{\langle u_r|v_2\rangle}{|v_2|^2} v_2 - \ldots\ldots - \frac{\langle u_r|v_{r-1}\rangle}{|v_{r-1}|^2} v_{r-1}}.$$
Multiply by a convenient factor to remove fractions.
(4) FOR r = 1 TO n, LET $\mathbf{w_r = \frac{v_r}{|v_r|}}$.

Example 13: Find an orthonormal basis for $V = \langle(1, 1, 1, 1), (1, 2, 3, 4), (1, -1, 1, 0)\rangle$.
Solution:

	u = basis	**v = orthogonal basis**	**|v|**	**w = orthonormal basis**
1	(1, 1, 1, 1)	(1, 1, 1, 1)	2	$\frac{1}{2}(1, 1, 1, 1)$
2	(1, 2, 3, 4)	(–3, –1, 1, 3)	$2\sqrt{5}$	$\frac{1}{2\sqrt{5}}(-3, -1, 1, 3)$
3	(1, –1, 1, 0)	(12, –26, 16, –2)	$6\sqrt{30}$	$\frac{1}{3\sqrt{30}}(6, -13, 8, -1)$

WORKING: $v_2 = u_2 - \frac{\langle u_2|v_1\rangle}{|v_1|^2} v_1 = (1, 2, 3, 4) - \frac{10}{4}(1, 1, 1, 1) = (1, 2, 3, 4) - \frac{5}{2}(1, 1, 1, 1)$.
Multiply by 2, so now $v_2 = (2, 4, 6, 8) - 5(1, 1, 1, 1) = (-3, -1, 1, 3)$.
$v_3 = u_3 - \frac{\langle u_3|v_1\rangle}{|v_1|^2} v_1 - \frac{\langle u_3|v_2\rangle}{|v_2|^2} v_2 = (1, -1, 1, 0) - \quad (1, 1, 1, 1) - \left(\frac{-1}{20}\right)(-3, -1, 1, 3)$.
Multiply by 20, so now $v_3 = (20, -20, 20, 0) - (5, 5, 5, 5) + (-3, -1, 1, 3) = (12, -26, 16, -2)$.

Example 14: Let V be the function space $\langle 1, x, x^2\rangle$ made into a Euclidean space by defining $\langle u(x)|v(x)\rangle = \int_0^1 u(x)v(x)dx$. Find an orthonormal basis for V.

Solution:

	u = basis	**v = orthogonal basis**	**|v|**	**w = orthonormal basis**
1	1	1	1	1
2	**x**	$2x - 1$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}\,(2x - 1)$
3	$\mathbf{x^2}$	$6x^2 - 6x + 1$	$\sqrt{\frac{1}{5}}$	$\sqrt{5}\,(6x^2 - 6x + 1)$

WORKING:

$$\langle u_2(x)|v_1(x)\rangle = \int_0^1 x\,dx = \left[\frac{1}{2}x^2\right]_0^1 = \frac{1}{2}.$$

$$|v_1(x)|^2 = \int_0^1 1dx = [x]_0^1 = 1$$

$v_2(x) = u_2(x) - \frac{\langle u_2(x)|v_1(x)\rangle}{|v_1(x)|^2} v_1(x) = x - \frac{1}{2}$.
Multiply by 2, so now $v_2(x) = 2x - 1$.

$$\langle u_3(x)|v_1(x)\rangle = \int_0^1 x^2 = \left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{3}.$$

$$\langle u_3(x)|v_2(x)\rangle = \int_0^1 x^2(2x-1)dx = \int_0^1 2x^3 - x^2\,dx = \left[\frac{1}{2}x^4 - \frac{1}{3}x^3\right]_0^1 = \frac{1}{6}.$$

$$|v_2(x)|^2 = \int_0^1 (2x-1)^2\,dx = \int_0^1 4x^2 - 4x + 1dx = \left[\frac{4}{3}x^3 - 2x^2 + x\right]_0^1 = \frac{1}{3}.$$

$$v_3(x) = u_3(x) - \frac{\langle u_3(x)|v_1(x)\rangle}{|v_1(x)|^2} v_1(x) - \frac{\langle u_3(x)|v_2(x)\rangle}{|v_2(x)|^2} v_2(x)$$
$$= x^2 - \frac{1}{3} - \frac{1/6}{1/3}(2x - 1)$$
$$= x^2 - \frac{1}{3} - \frac{1}{2}(2x - 1)$$

Multiply by 6, so now $v_3(x) = 6x^2 - 2 - 3(2x - 1)$
$= 6x^2 - 6x + 1$.

$$|v_3(x)|^2 = \int_0^1 (6x^2 - 6x + 1)^2\,dx = \int_0^1 36x^4 - 72x^3 + 48x^2 - 12x + 1dx = \left[\frac{36}{5}x^5 - 18x^4 + 16x^3 - 6x^2 + x\right]_0^1 = \frac{1}{5}$$

§3.4. Fourier Series

The most important applications of inner product spaces involves function spaces with the inner product defined by means of an integral. Fourier series are infinite series in an infinite dimensional function space. However it is not appropriate here to give more than a cursory overview because to discuss them properly requires not only a good knowledge of integration, but a deep understanding of the convergence of infinite series.

For any positive integer n the functions 1, cos nx and sin nx are periodic, with period 2π. [This does not mean that they do not have smaller period, but simply that for each of them
$f(x + 2\pi) = f(x)$ for all x.]
Take the space T spanned by all of these functions.

$$\text{So } T = \langle 1, \cos x, \cos 2x, \ldots, \sin x, \sin 2x, \ldots\rangle.$$

Define the inner product on T as $\langle u(x) \mid v(x)\rangle = \int_0^{2\pi} u(x)v(x)\,dx$. T is an infinite dimensional vector space. Clearly, for every function $f(x) \in T$, $f(x + 2\pi) = f(x)$. If f(x) is a continuous function for which $f(x + 2\pi) = f(x)$ we may ask whether $f(x) \in T$. The answer is usually no. Such an f(x) may not be a linear combination of 1, cos x, cos 2x, ..., sin x, sin 2x, ... But remember that a linear combination is a *finite* linear combination. It may well be that f(x) can be expressed as an *infinite* series involving these functions. That is we might have $f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$. Such a series is called a **Fourier series**, named after the French mathematician Joseph de Fourier [1768 – 1830]. Of course, for this to make sense we would need this series to converge, which why we need to know a lot about infinite series in order to study Fourier series.

But suppose we limit the values of n. Let $T_N = \langle 1, \sin x, \sin 2x, \ldots, \cos x, \cos 2x, \ldots\rangle$. We can show that these 2N + 1 functions are linearly independent. In fact, they are mutually orthogonal. So T_N is a 2N + 1 dimensional Euclidean space.

For n > 0, $|\cos nx|^2 = \int_0^{2\pi} \cos^2 nx\,dx = \pi$ and $|\sin x|^2 = \int_0^{2\pi} \sin^2 nx\,dx = \pi$. Clearly $|1|^2 = \int_0^{2\pi} 1\,dx = 2\pi$.
(Remember that |1| here is not the absolute value but rather the length of the function 1.)
By theorem 5, if $F(x) = a_0 + \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx)$ then $a_0 = \dfrac{\langle F(x)|1\rangle}{|1|^2} = \dfrac{1}{2\pi}\int_0^{2\pi} F(x)\,dx$ and,
if n > 0, $a_n = \dfrac{\langle F(x)|\cos nx\rangle}{|\cos nx|^2} = \dfrac{1}{\pi}\int_0^{2\pi} F(x)\cos nx\,dx$ and $b_n = \dfrac{\langle F(x)|\sin nx\rangle}{|\sin nx|^2} = \dfrac{1}{\pi}\int_0^{2\pi} F(x)\sin nx\,dx$.

A function in T_N must be continuous and have period 2π. But by no means does every such function belong to T_N. However if F(x) is continuous and has period 2π then it can be approximated by a function in T_N, with the approximation getting better as N becomes larger. Even functions with period 2π having some discontinuities can be so approximated. (We won't go into details here as to the precise conditions, or how close the approximation will be.)

If $F^{(N)}(x)$ is the approximation to F(x) in T_N then $F^{(N)}(x) = a_0 + \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx)$ where $a_0 = \dfrac{1}{2\pi}\int_0^{2\pi} F^{(N)}(x)\,dx$ and, if n > 0, $a_n = \dfrac{1}{\pi}\int_0^{2\pi} F^{(N)}(x)\cos nx\,dx$ and $b_n = \dfrac{1}{\pi}\int_0^{2\pi} F^{(N)}(x)\sin nx\,dx$.
Now it can be shown that generally these integrals can be approximated by the corresponding integrals with $F^{(N)}$ replaced by F(x). Letting $N \to \infty$ it can be shown that if F[x] can be expressed as a Fourier series $a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ then $a_0 = \dfrac{1}{2\pi}\int_0^{2\pi} F(x)\,dx$ and, if n > 0,

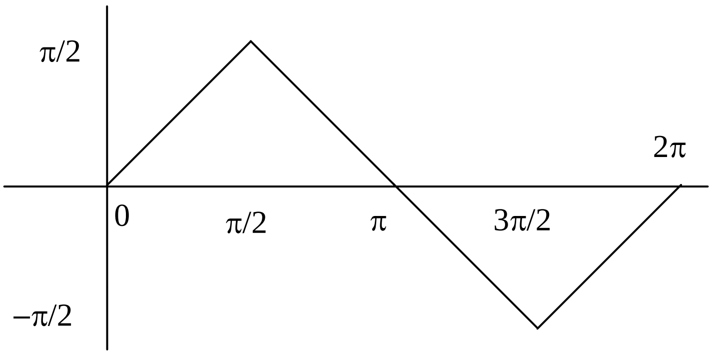
$a_n = \frac{1}{\pi}\int_0^{2\pi} F(x)\cos nx\,dx$ and $b_n = \frac{1}{\pi}\int_0^{2\pi} F(x)\sin nx\,dx$.

Example 15: Find the Fourier series for the function

$$\left.\begin{array}{c} F(x) = x \text{ if } 0 \le x \le \pi/2 \\ F(x) = \pi - x \text{ if } \pi/2 \le x \le 3\pi/2 \\ F(x) = x - 2\pi \\ F(x + 2\pi) = F(x) \text{ for all } x \end{array}\right\}$$

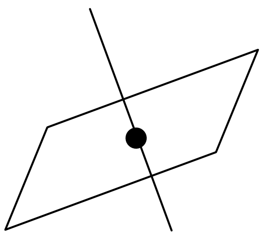
Answer: The solution involves a fair bit of integration by parts and, since this is not a calculus course, we omit the details and simply give the answer.

$$F(x) = \frac{4}{\pi}\left[\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \cdots\cdots\cdots\right].$$



§3.5. Orthogonal Complements

In $\mathbf{R}^3$ the normal to a plane through the origin is a plane through the origin. Every vector in the plane is orthogonal (i.e. perpendicular if they are non-zero) to every vector along the line. The line and the plane are said to be orthogonal complements of one another.



The **orthogonal complement** of a subspace U, of V is defined to be:

$$\mathbf{U}^{\perp} = \{v \in V \mid \langle u \mid v\rangle = 0 \text{ for all } u \in U\}.$$

Intuitively it would seem that $U^{\perp\perp}$ should be U, but there are examples where this is not so. However for finite-dimensional subspaces it *is* true. This follows from the following important theorem.

Theorem 7: If U is a finite-dimensional subspace of the vector space then $U^{\perp}$ is also subspace of V and $V = U \oplus U^{\perp}$.

Proof: (1) $U^{\perp}$ is a subspace of V.

Let v, w $\in U^{\perp}$ and let u ∈ U. Then $\langle u \mid v + w\rangle = \langle u \mid v\rangle + \langle u \mid w\rangle = 0 + 0 = 0$.

Hence $v + w \in U^T$.

Let $v \in U^{\perp}$ and let λ be a scalar. Then $\langle u \mid \lambda v\rangle = \bar{\lambda}\langle u \mid v\rangle = \bar{\lambda}.0 = 0$.

Hence $\lambda v \in U^{\perp}$ and so we have shown that $U^{\perp}$ is a subspace.

(2) $U \cap U^{\perp} = 0$.

Suppose $v \in U \cap U^{\perp}$. Then v is orthogonal to itself, and so $\langle v \mid v\rangle = 0$. By the axioms of an inner product space this implies that $v = 0$.

(3) $V = U + U^{\perp}$.

Let $u_1, \ldots u_n$ be an orthonormal basis for U.

Let $v \in V$, $u = \langle v \mid u_1\rangle u_1 - \ldots - \langle v \mid u_n\rangle u_n$ and let $w = v - u$.

Then for each i, $\langle w \mid u_i\rangle = \langle v \mid u_i\rangle - \langle v \mid u_i\rangle\langle u_i \mid u_i\rangle$ since $\langle u_j \mid u_i\rangle = 0$ if $i \neq j$

$= \langle v \mid u_i\rangle - \langle v \mid u_i\rangle$ since $\langle u_i \mid u_i\rangle = 1$

$= 0$.

Hence $w \in U^{\perp}$. Clearly $u \in U$. So $v = u + w \in U + U^{\perp}$.

Theorem 8: If U is a subspace of a finite-dimensional vector space then $U^{\perp\perp} = U$.

Proof: Suppose $u \in U$ and let $v \in U^{\perp}$. Then $\langle u \mid v\rangle = 0$. Hence $\langle v \mid u\rangle = 0$. Since this holds for all $v \in U^{\perp}$, and so $u \in U^{\perp\perp}$. So it follows that $U \leq U^{\perp\perp}$.

Now $V = U \oplus U^{\perp} = U^{\perp} \oplus U^{\perp\perp}$ so $\dim U = \dim U^{\perp\perp}$. Hence $U = U^{\perp\perp}$.

EXERCISES FOR CHAPTER 3

Exercise 1: If $v_1 = (x_1, y_1)$ and $v_2 = (x_2, y_2)$ define $\langle u \mid v\rangle = 2x_1x_2 - 2x_1y_2 - 2x_2y_1 + 5y_1y_2$ and $[u \mid v] = 2x_1x_2 + 2x_1y_2 + 2x_2y_1 + y_1y_2$. Show that under one of these products $\mathbf{R}^2$ becomes a Euclidean space and under the other it is not a Euclidean space.

Exercise 2: Find an orthonormal basis for $\langle(2, 2, 1), (3, 1, -5)\rangle$.

Exercise 3: Find an orthonormal basis for $\langle(1, 1, 0, 1, 1), (1, 0, 1, 1, 1), (1, 1, 1, 0, 1), (1, 1, 1, 1, 0)\rangle$.

Exercise 4: Find an orthonormal basis for the function space $\langle 1, \sqrt{x}, x\rangle$ where

$$\langle u(x) \mid v(x)\rangle = \int_0^1 u(x)v(x)\,dx .$$

Exercise 5: Find an orthonormal basis for the function space $\langle 1, 2x, \sin x\rangle$ where

$$\langle u(x) \mid v(x)\rangle = \int_0^{\pi/2} u(x)v(x)\,dx .$$

Exercise 6: Find the orthogonal complement for $\langle(1, 3, 6), (2, 1, 2)\rangle$ in $\mathbf{R}^3$.

Exercise 7: Find the orthogonal complement of $\langle(1, 1, 1, 1), (1, 0, 1, 0)\rangle$ in $\mathbf{R}^4$.

Exercise 8: Find the orthogonal complement of $\langle 1, x\rangle$ in the vector space $\langle 1, x, x^2\rangle$, where $\langle u(x) \mid v(x)\rangle$ is defined to be $\int_0^1 u(x)v(x)\,dx$.

SOLUTIONS FOR CHAPTER 3

Exercise 1: Axioms (1), (2), (3) are easily checked for both products. The simplest way to check them is to let $A = \begin{pmatrix} 2 & -2 \\ -2 & 3 \end{pmatrix}$ and $B = \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix}$. Then $\langle u \mid v\rangle = uAv^T$ and $[u \mid v] = uBv^T$. It is now very simple to check these first three axioms.

(4) If v = (x, y) then $\langle v \mid v\rangle = 2x^2 - 4xy + 5y^2 = 2(x - y)^2 + 3y^2 \geq 0$ for all x, y.

If v = (x, y) then $[v \mid v] = 2x^2 + 4xy + y^2 = 2(x + y)^2 - y^2$. When x = 1 and y = – 1 this is negative. Hence under the product [u | v] R^2 *is not* a Euclidean space.

(5) If $\langle v \mid v\rangle = 0$ then x = y and y = 0 so v = 0.

Hence under the product $\langle v \mid v\rangle$, $\mathbf{R}^2$ *is* a Euclidean space.

Exercise 2:

	u = basis	**v = orthogonal basis**	**|v|**	**w = orthonormal basis**
1	(2, 2, 1)	(2, 2, 1)	3	$\frac{1}{3}(2, 2, 1)$
2	(3, 1, –5)	(1, –1, –6)	$\sqrt{38}$	$\frac{1}{\sqrt{38}}(1, -1, -6)$

WORKING: $v_2 = u_2 - \frac{\langle u_2|v_1\rangle}{|v_1|^2} v_1 = (3, 1, -5) - \frac{3}{3}(2, 2, 1) = (3, 1, -5) - (2, 2, 1) = (1, -1, -6)$.

Exercise 3:

	u = basis	**v = orthogonal basis**	**|v|**	**w = orthonormal basis**
1	(1, 1, 0, 1, 1)	(1, 1, 0, 1, 1)	2	$\frac{1}{2}(1, 1, 0, 1, 1)$
2	(1, 0, 1, 1, 1)	(1, –3, 4, 1, 1)	$\sqrt{28}$	$\frac{1}{\sqrt{28}}(1, -3, 4, 1, 1)$
3	(1, 1, 1, 0, 1)	(1, 4, 4, –6, 1)	$\sqrt{70}$	$\frac{1}{\sqrt{70}}(1, 4, 4, -6, 1)$
4	(1, 1, 1, 1, 0)	(–91, –49, 56, –49, 231)	$\sqrt{69580}$	$\frac{1}{\sqrt{69580}}(-91, -49, 56, -49, 231)$

WORKING: $v_2 = u_2 - \frac{\langle u_2|v_1\rangle}{|v_1|^2} v_1 = (1, 0, 1, 1, 1) - \frac{3}{4}(1, 1, 0, 1, 1)$.

Multiply by 4. Now $v_2 = (4, 0, 4, 4, 4) - (3, 3, 0, 3, 3) = (1, -3, 4, 1, 1)$.

$v_3 = u_3 - \frac{\langle u_3|v_1\rangle}{|v_1|^2} v_1 - \frac{\langle u_3|v_2\rangle}{|v_2|^2} v_2 = (1, 1, 1, 0, 1) - \frac{3}{4}(1, 1, 0, 1, 1) - \frac{3}{28}(1, -3, 4, 1, 1)$.

Multiply by 28. Now $v_3 = (28, 28, 28, 0, 28) - (21, 21, 0, 21, 21) - (3, -9, 12, 3, 3)$
$= (4, 16, 16, -24, 4)$.

Divide by 4. Now $v_3 = (1, 4, 4, -6, 1)$.

$v_4 = u_4 - \frac{\langle u_4|v_1\rangle}{|v_1|^2} v_1 - \frac{\langle u_4|v_2\rangle}{|v_2|^2} v_2 - \frac{\langle u_4|v_3\rangle}{|v_3|^2} v_3 = (1, 1, 1, 1, 0) - \frac{3}{2}(1, 1, 0, 1, 1) - \frac{3}{28}(1, -3, 4, 1, 1)$
$- \frac{3}{70}(1, 4, 4, -6, 1)$.

Multiply by 140. Now $v_4 = (140, 140, 140, 140, 0) - (210, 210, 0, 210, 210) - (15, -45, 60, 15, 15)$
$- (6, 24, 24, -36, 6)$
$= (-91, -49, 56, -49, 231)$.

Exercise 4:

	u = basis	v = orthogonal basis	|v|	w = orthonormal basis
1	1	1	1	1
2	$\sqrt{x}$	$3\sqrt{x} - 2$	$\frac{1}{\sqrt{2}}$	$\sqrt{2}(3\sqrt{x} - 2)$
3	x	$10x - 12\sqrt{x} + 3$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}(10x - 12\sqrt{x} + 3)$

WORKING:

$$\langle u_2(x)|v_1(x)\rangle = \int_0^1 \sqrt{x}\,dx = \left[\frac{2}{3}x^{3/2}\right]_0^1 = \frac{2}{3}.$$

$$v_2(x) = u_2(x) - \frac{\langle u_2(x)|v_1(x)\rangle}{|v_1(x)|^2} v_1(x) = \sqrt{x} - \frac{2/3}{1}.1 = \sqrt{x} - \frac{2}{3}.$$

Multiply by 3, so now $v_2(x) = 3\sqrt{x} - 2$.

$$|v_2(x)|^2 = \int_0^1 (3\sqrt{x} - 2)^2\,dx$$

$$= \int_0^1 (9x - 12\sqrt{x} + 4)dx$$

$$= \left[\frac{9}{2}x^2 - 8x^{3/2} + 4x\right]_0^1$$

$$= \frac{9}{2} - 8 + 4$$

$$= \frac{1}{2}.$$

$$\langle u_3(x)|v_1(x)\rangle = \int_0^1 x\,dx$$

$$= \left[\frac{1}{2}x^2\right]_0^1$$

$$= \frac{1}{2}.$$

$$\langle u_3(x)|v_2(x)\rangle = \int_0^1 x(3\sqrt{x} - 2)dx$$

$$= 3\int_0^1 x^{3/2}\,dx - 2\int_0^1 x\,dx$$

$$= 3\left[\frac{2}{5}x^{5/2}\right]_0^1 - 2\left[\frac{1}{2}x^2\right]_0^1$$

$$= \frac{6}{5} - 1$$

$$= \frac{1}{5}.$$

$$v_3(x) = u_3(x) - \frac{\langle u_3(x)|v_1(x)\rangle}{|v_1(x)|^2} v_1(x) - \frac{\langle u_3(x)|v_2(x)\rangle}{|v_2(x)|^2} v_2(x)$$

$$= x - \frac{1/2}{1}.1 - \frac{1/5}{1/2}(3\sqrt{x} - 2)$$

$= x - \frac{1}{2} - \frac{2}{5}(3\sqrt{x} - 2)$

$= x - \frac{1}{2} - \frac{6}{5}\sqrt{x} + \frac{4}{5}$

$= x - \frac{6}{5}\sqrt{x} + \frac{3}{10}$

.

Multiplying by 10 we now take $v_3(x) = 10x - 12\sqrt{x} + 3$.

$$|v_3(x)|^2 = \int_0^1 \left(10x - 12\sqrt{x} + 3\right)^2 dx$$

$$= \int_0^1 \left(100x^2 + 144x + 9 - 240x^{3/2} + 60x - 72\sqrt{x}\right)dx$$

$$= \int_0^1 \left(100x^2 + 204x + 9 - 240x^{3/2} - 72\sqrt{x}\right)dx$$

$$= \left[\frac{100}{3}x^3 + 102x^2 + 9x - 96x^{5/2} - 48x^{3/2}\right]_0^1$$

$$= \frac{100}{3} + 102 + 9 - 96 - 48$$

$$= \frac{100}{3} - 33$$

$$= \frac{1}{3}.$$

Exercise 5:

	u = basis	**v = orthogonal basis**	**|v|**	**w = orthonormal basis**
1	1	1	$\sqrt{\frac{\pi}{2}}$	$\sqrt{\frac{2}{\pi}}$
2	2x	$2x - \pi$	$\sqrt{\frac{\pi^3 - 3\pi^2 + 3\pi}{6}}$	$\sqrt{\frac{6}{\pi^3 - 3\pi^2 + 3\pi}}\ (2x - \pi)$
3	sin x	$\pi \sin x - 2$	$\frac{\sqrt{\pi^3 - 8\pi}}{2}$	$\frac{2}{\sqrt{\pi^3 - 8\pi}}(\pi \sin x - 2)$

WORKING:

$$|v_1(x)|^2 = \int_0^{\pi/2} 1\,dx = [x]_0^{\pi/2} = \frac{\pi}{2}$$

$$\langle u_2(x)|v_1(x)\rangle = \int_0^{\pi/2} 2x\,dx = [x^2]_0^{\pi/2} = \frac{\pi^2}{4}$$

$$v_2(x) = u_2(x) - \frac{\langle u_2(x)|v_1(x)\rangle}{|v_1(x)|^2} v_1(x) = x - \frac{\pi^2/4}{\pi/2}.1 = x - \frac{\pi}{2}.$$

Multiply by 2, so now $v_2(x) = 2x - \pi$.

$$|v_2(x)|^2 = \int_0^{\pi/2} (2x - \pi)^2\,dx$$

$$= \int_0^{\pi/2} \left(4x^2 - 4\pi x + \pi^2\right)dx$$

$$= \left[\frac{4}{3}x^3 - 2\pi x^2 + \pi^2 x\right]_0^{\pi/2}$$

$$= \frac{1}{6}\pi^3 - \frac{1}{2}\pi^2 + \frac{1}{2}\pi$$

$$= \frac{1}{6}\left(\pi^3 - 3\pi^2 + 3\pi\right).$$

$$\langle u_3(x) | v_1(x)\rangle = \int_0^{\pi/2} \sin x \, dx$$

$$= [-\cos x]_0^{\pi/2}$$

$$= 1.$$

$$\langle u_3(x) | v_2(x)\rangle = \int_0^{\pi/2} (2x - \pi)\sin x \, dx$$

$$= \int_0^{\pi/2} 2x \sin x \, dx \; - \int_0^{\pi/2} \pi \sin x \, dx$$

$$= [-2x\cos x]_0^{\pi/2} + 2\int_0^{\pi/2} \cos x \, dx \; - \int_0^{\pi/2} \pi \sin x \, dx$$

$$= 0 + 2[\sin x]_0^{\pi/2} + \pi[\cos x]_0^{\pi/2}$$

$$= \pi - \pi$$

$$= 0.$$

$$v_3(x) = u_3(x) - \frac{\langle u_3(x) | v_1(x)\rangle}{|v_1(x)|^2} v_1(x) - \frac{\langle u_3(x) | v_2(x)\rangle}{|v_2(x)|^2} v_2(x)$$

$$= \sin x - \frac{2}{\pi}.1 - 0.(\pi\cos x - \sin x)$$

$$= \sin x - \frac{2}{\pi}.$$

Multiplying by π we now take $v_3(x) = \pi \sin x - 2$.

$$|v_3(x)|^2 = \int_0^{\pi/2} (\pi \sin x - 2)^2 \, dx$$

$$= \int_0^{\pi/2} \left(\pi^2 \sin^2 x - 4\pi \sin x + 4\right) dx$$

$$= \pi^2 \int_0^{\pi/2} \sin^2 dx \; - 4\pi \int_0^{\pi/2} \sin x \, dx + 4\int_0^{\pi/2} dx$$

$$= \frac{\pi^2}{2} \int_0^{\pi/2} (1 - \cos 2x) dx \; - 4\pi \int_0^{\pi/2} \sin x \, dx + 4\int_0^{\pi/2} dx$$

$$= \frac{\pi^2}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi/2} + 4\pi[\cos x]_0^{\pi/2} + 2\pi$$

$$= \frac{\pi^2}{2}\left[\frac{\pi}{2} - 0\right] - 4\pi + 2\pi$$

$= \dfrac{\pi^3}{4} - 2\pi$

$= \dfrac{1}{4}\left(\pi^3 - 8\pi\right).$

Exercise 6:
First Solution: Let $\mathbf{u_1}$ = (2, 1, 2) and $\mathbf{u_2}$ = (2, 3, 6)〉. Suppose (x, y, z) is orthogonal to both $\mathbf{u_1}$ and $\mathbf{u_2}$. Then 2x + y + 2z = 0 and 2x + 3y + 6z = 0.

$\begin{pmatrix} 2 & 1 & 2 \\ 2 & 3 & 6 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 1 & 2 \\ 0 & 2 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}$ so x= 0, z = k, y = –2k.

Hence the orthogonal complement is 〈(0, – 2, 1)〉.

Second Solution: We could take, as a third vector (1, 1, 1), being outside of the space spanned by **a** and **b**, and use the Gram Schmidt process. However we are content with an *orthogonal* basis.

	u = basis	**v = orthogonal basis**	$\mathbf{\lvert v\rvert^2}$
1	(2, 1, 2)	(2, 1, 2)	9
2	(2, 3, 6)	(–5, 2, 4)	45
3	(1, 1, 1)	(0, 2, –1)	

WORKING:

$\mathbf{v_2} = \mathbf{u_2} - \dfrac{\langle \mathbf{u_2} \mid \mathbf{v_1} \rangle}{|\mathbf{v_1}|^2}\mathbf{v_1}$

$= (2, 3, 6) - \dfrac{19}{9}(2, 1, 2).$

Multiply by 9 to get the new v_2 to be v_2 = 9(2, 3, 6) – 19(2, 1, 2)
= (18, 27, 54) – (38, 19, 38)
= (–20, 8, 16).

Perhaps it would now be a good idea to divide by 4 to get a new $\mathbf{v_2}$ as $\mathbf{v_2}$ = (–5, 2, 4).

$\langle \mathbf{u_3} \mid \mathbf{v_1} \rangle = 5$ and $\langle \mathbf{u_3} \mid \mathbf{v_2} \rangle = 1$.

$\mathbf{v_3} = \mathbf{u_3} - \dfrac{\langle \mathbf{u_3} \mid \mathbf{v_1} \rangle}{|\mathbf{v_1}|^2}\mathbf{v_1} - \dfrac{\langle \mathbf{u_3} \mid \mathbf{v_2} \rangle}{|\mathbf{v_2}|^2}\mathbf{v_2}$

$= (1, 1, 1) - \dfrac{5}{9}(2, 1, 2) - \dfrac{1}{45}(-5, 2, 4)$

Multiply by 45 to get a new $\mathbf{v_3}$ as $\mathbf{v_3}$ = (45, 45, 45) – (50, 25, 50) – (–5, 2, 4)
= (0, 18, –9).

Divide by 9 to get a new $\mathbf{v_3}$ as $\mathbf{v_3}$ = (0, 2, –1).

Hence the orthogonal complement is 〈(0, 2, –1)〉.

Third Solution: A third method, that only works for $\mathbf{R}^3$, is to simply find $\mathbf{u_1} \times \mathbf{u_2}$.

$u_1 \times u_2 = \begin{vmatrix} i & j & k \\ 2 & 1 & 2 \\ 2 & 3 & 6 \end{vmatrix} = (6 - 6)\mathbf{i} - (12 - 4)\mathbf{j} + (6 - 2)\mathbf{k} = (0, -8, 4)$. So the orthogonal complement is 〈(0, –8, 4)〉 = 〈(0, 2, –1)〉.

You can make up your own mind as to which is the easiest method!

Exercise 7: Here we cannot use the vector product.
Suppose (x, y, z, w) is orthogonal to both vectors. Then we have a system of two homogeneous linear equations that is represented by

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

So w = h, z = k, y = –h, x = –k giving the vector (–k, –h, k, h).
Taking h = 1, k = 0 and h = 0, k = 1, we get a basis for the orthogonal complement, which is $\langle(0, -1, 0, 1), (-1, 0, 1, 0)\rangle$.

Exercise 8: $\int_0^1 \left(a + bx + cx^2\right).1\,dx = \left[ax + \frac{b}{2}x^2 + \frac{c}{3}x^3\right]_0^1 = a + \frac{b}{2} + \frac{c}{3}$ and

$\int_0^1 \left(a + bx + cx^2\right).x\,dx = \left[\frac{a}{2}x^2 + \frac{b}{3}x^3 + \frac{c}{4}x^4\right]_0^1 = \frac{a}{2} + \frac{b}{3} + \frac{c}{4}$.

Hence $w(x) = a + bx + cx^2$ is orthogonal to both 1 and x if $a + \frac{b}{2} + \frac{c}{3} = 0$ and $\frac{a}{2} + \frac{b}{3} + \frac{c}{4} = 0$.

We solve the homogeneous system $\begin{pmatrix} 6 & 3 & 2 \\ 6 & 4 & 3 \end{pmatrix} \to \begin{pmatrix} 6 & 3 & 2 \\ 0 & -1 & 1 \end{pmatrix}$.

This gives c = k, b = k, 6a = –5k.
Take k = 6. Then a = –5, b = 6, c = 6 and hence $w(x) = -5 + 6x + 6x^2$.
Hence the orthogonal complement is $\langle 6x^2 + 6x - 5\rangle$.